7 INTRODUCTION TO PLATE BUCKLING

1. 0 INTRODUCTION

Steel plates are widely used in buildings, bridges, automobiles and ships. Unlike beams and columns, which have lengths longer than the other two dimensions and so are modeled as linear members, steel plates have widths comparable to their lengths and so are modeled as two-dimensional plane members.

Just as long slender columns undergo instability in the form of buckling, steel plates under membrane compression also tend to buckle out of their plane. The buckled shape depends on the loading and support conditions in both length and width directions.

However, unlike columns, plates continue to carry loads even after buckling in a stable manner. Their post-buckling strengths, especially in the case of slender plates, can thus be substantially greater than the corresponding buckling strengths. This property is of great interest to structural engineers as it can be utilized to their advantage.

In this chapter, the expression for the critical buckling strength, of a flat plate simply supported on all four sides, is derived. The post-buckling behaviour of plates is described in terms of both stability and strength and compared with the post-buckling behaviour of a column. The concept of effective width is introduced to tackle the non-uniform distribution of stress in practical plates before and after buckling.

Buckling of web plates in shear is described and an expression to calculate their ultimate capacity is also given for use in design. A plate buckled in shear, can also carry additional shear due to the tension field action. Interaction formulas for plates under various load combinations are also given.

2.0 CRITICAL STRESS FOR PLATE BUCKLING

2.1 Rectangular flat plate simply supported on four sides

Consider a rectangular perfectly flat plate simply supported on all four sides and subjected to uniform compressive force N_x per unit length in the x-direction (Fig.1). The equilibrium equation for such a plate is given by

$$\frac{\partial^4 w}{\partial x^4} + \frac{2\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{12(1-\nu^2)}{Et^3}\left(-N_x \frac{\partial^2 w}{\partial x^2}\right) \tag{1}$$

where, w denotes the deflection in the z-direction of any point (x,y).

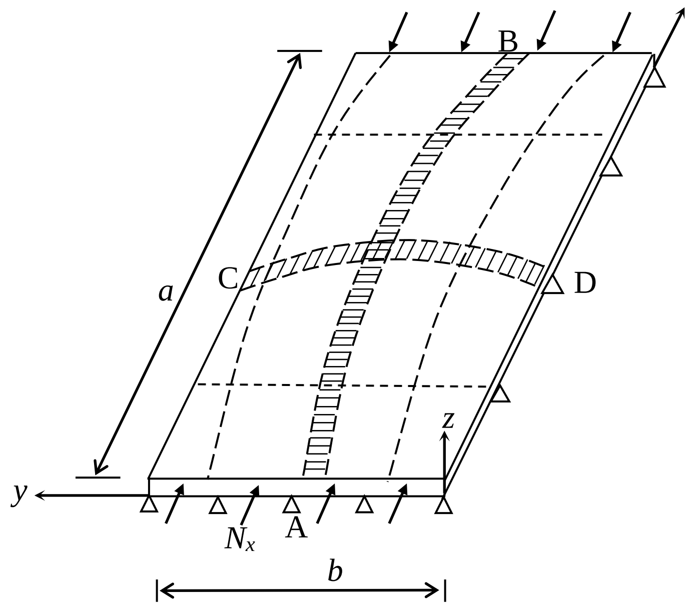

Fig.1 Buckling of Plate under Uni-axial Compression

w can be assumed as

$$w = \sum_{m=1,2,3,\ldots} \sum_{n=1,2,3,\ldots} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{2}$$

The *m* and *n* in Eq. 2, indicate the number of half sine waves in the buckled mode. It may be noted that this assumed shape automatically satisfies the hinged boundary conditions for the plate, that is $w = 0$ at $x = 0$, $x = a$, $y = 0$ and $y = b$.
Substitution of Eq. (2) in Eq. (1) gives

$$\left(\frac{m^4\pi^4}{a^4} + 2\frac{m^2n^2\pi^4}{a^2b^2} + \frac{n^4\pi^4}{b^4}\right) = \frac{12(1-\nu^2)}{Et^3}(N_x)_{cr}\frac{m^2\pi^2}{a^2} \tag{3}$$

Therefore,

$$(N_x)_{cr} = \frac{\pi^2Et^3}{12(1-\nu^2)}\frac{\left(m^2/a^2 + n^2/b^2\right)^2}{m^2/a^2} = \frac{\pi^2Et^3}{12(1-\nu^2)}\left(\frac{m}{a} + \frac{n^2a}{mb^2}\right)^2 \tag{4}$$

The lowest value of the membrane buckling stress $(N_x)_{cr}$, in Eq. (4) is obtained for $n=1$ and can also be written as follows,

$$(N_x)_{cr} = \frac{\pi^2Et^3}{12(1-\nu^2)b^2}\left(m\frac{b}{a} + \frac{1}{m}\frac{a}{b}\right)^2 \tag{5}$$

Denoting the quantity within larger brackets by k and noting that the buckling load, N_{cr}, is the product of the buckling stress σ_{cr} and the thickness, we get the buckling stress as

$$\sigma_{cr} = \frac{k\pi^2 E}{12(1-\nu^2)(b/t)^2} \tag{6}$$

The expression for the critical buckling stress is similar to the Euler stress for columns [$\sigma_e = \pi^2 E/(\ell/r)^2$] except for the fact that it is a function of the width-thickness ratio b/t. Why should the critical buckling stress in the x-direction be a function of the width b in the y-direction ?

As the compressive load N_x on the plate is increased and reaches the critical buckling load N_{cr}, the central part of the plate such as the strip *AB* tends to buckle. Now, if we consider a transverse strip *CD*, we can realize that this strip resists the tendency of the strip *AB* to deflect out of the plane of the plate (z-direction). The shorter the width b, more will be the resistance offered by *CD* to *AB*. Hence the strip *AB* until buckling behaves like a column on elastic foundation, whose stiffness depends on b. This is the reason why the width b figures in the expression for critical buckling stress.

Next, let us consider the influence of the length 'a' of the plate on the buckling shape. Equation (5) is plotted in Fig. 3, showing the variation of buckling strength with respect to b/a ratio and for various values of m.

Consider a plate whose length a is much greater than the width b. If a longitudinal strip such as *AB* in Fig. 1, tends to form a single buckle, its curvature will be much less than the curvature of the transverse strip *CD* which tries to resist the buckling. This means that the resistance is greater than the tendency to buckle and the strength corresponding to this mode ($m=1$) is very high. Therefore, the plate prefers to buckle such that the curvatures of longitudinal and transverse strips are as equal as possible. This leads to multiple buckles in alternate directions as shown in Fig.2 such that the buckles are as square as possible. If $a = 2b$, the plate develops two buckles, if $a = 3b$, it develops three buckles and so on (Fig. 2).

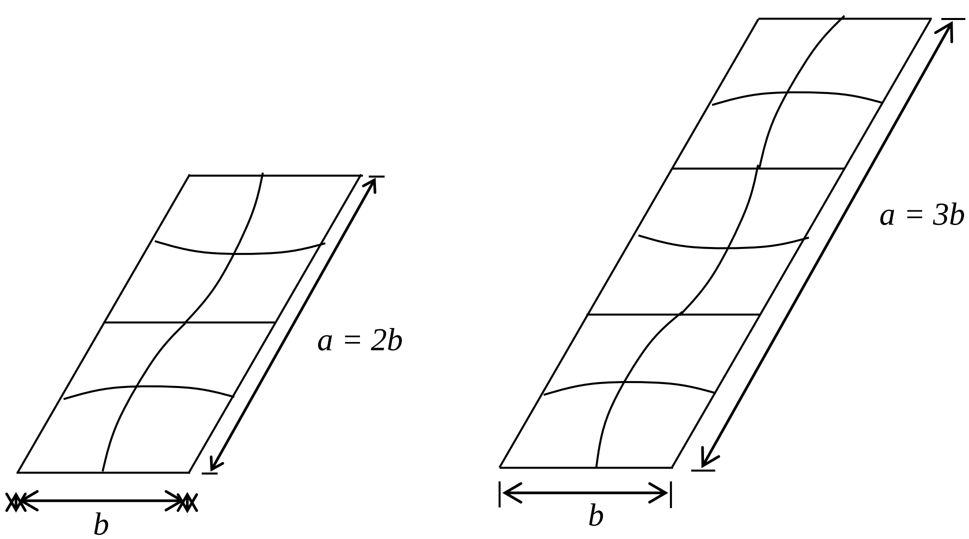

Fig. 2 Buckling Modes for Long Plates

Variation of *k*, the plate buckling coefficient, with aspect ratio (the ratio of the length, *a*, to the width, *b*) is shown in Fig. 3 for *m=1,2,3*, etc. It can be seen that the lowest value of the buckling coefficient is obtained for integral values of the aspect ratio. Correspondingly square half waves are the buckling mode shapes. Usually the plates are long in practice and for large aspect ratios the buckling coefficient is almost independent of the aspect ratio and is equal to the lowest value of 4.0. Hence the local buckling coefficient is taken to be the smallest value, independent of the aspect ratio and equals 4.0 for the case discussed.

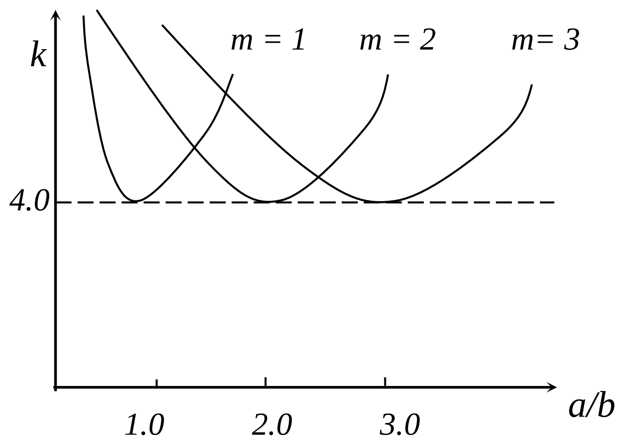

Fig. 3 k-values for a Simply Supported Plate

2.2 Plates with Other support Conditions

So far, it has been assumed that the plate is free to rotate about the longitudinal edges. Other edge conditions are of course possible. Consider for example, a box column made up of four plates as shown in Fig. 4 (*a*). If the flanges are relatively stiff, they would prevent the rotation of the corners [Fig. 4 (*b*)] and the web plate will behave as if its longitudinal edges are fixed. In this case, the bending resistance offered by the transverse strips such as *CD* will be considerably more than that of a plate with simply supported longitudinal edges and the buckling stress will be larger. If the flanges are also prone to buckling, then the corners will rotate as shown in Fig. 4c and the critical buckling stress will be the same as that for a plate with simply supported longitudinal edges. Therefore, the buckling coefficient is a function of the support condition along the longitudinal edges and the type of loading. It can be shown that the expression for the critical buckling coefficient is still valid except for the fact that the *k* values will be different. The *k* values for various common support conditions and loading cases are given in Table 1. Additional information can be found in Bulson (1970) and Timoshenko and Gere (1961).

In many rolled sections such as I-sections or channel sections, we find that flanges are similar to plates having one longitudinal edge simply supported and the other free. These are called *outstands* as against plate elements having both longitudinal edges simply supported (*internal elements*). From Table 2, we find that *k* value for outstands is *0.425* which is roughly one-tenth that for internal elements (*k=4*). The reason for such a low value is that the transverse strip (such as *CD* in Fig. 1) simply rotates and offers little bending resistance as shown in Fig. 4 (*d*).

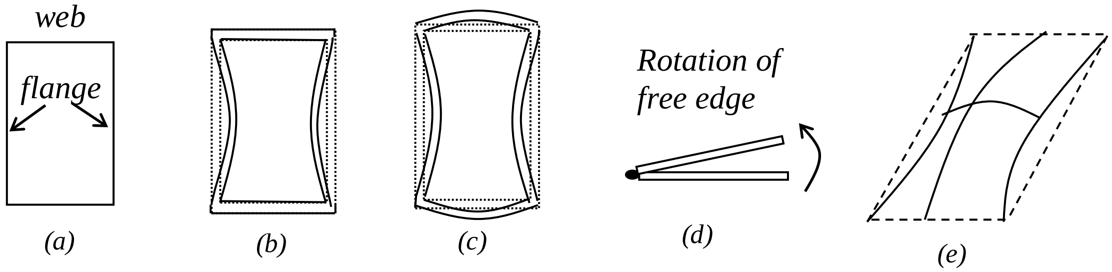

Fig. 4 Plate Elements with Different Edge Conditions

Table 1 Values of k for Different Load and Support Conditions

Load Condition	**Support Condition**	**Buckling Coefficient, *k***
Uniaxial Compressive Stress (σ_x)	Hinged-hinged	4.00
	Fixed-fixed	6.97
	Hinged-free	1.27
	Fixed free	0.43
Shear Stress (τ_{xy})	Hinged-hinged	5.35
	Fixed-fixed	8.99

The edge conditions not only affect the critical buckling stress but also influence the post-buckling behaviour. For a plate with longitudinal edges (edges parallel to the *x*-axis in Fig. 1) constrained to remain straight in the plane of the plate, the transverse stresses in the strip *CD* will be tensile and has the effect of stiffening the plate against lateral deflection. However, if the longitudinal edges are free to pull-in in the *y*-direction [Fig. 4(e)], there will be no transverse stresses in the strip *CD* and the plate will be less stiff compared to the previous case.

To ensure that a plate with a given support conditions fails by yielding rather than buckling, the corresponding critical buckling stress should be greater than the yield stress. Equating the expression given in Eq.(6) to the yield stress, the limiting value of the width-thickness ratio to ensure yielding before plate buckling can be obtained as

$$\left(\frac{b_{\lim}}{t}\right) \le \left(\frac{k\pi^2 E}{12(1-\nu^2)\, f_y}\right)^{1/2} \qquad (7)$$

The codes prescribe different limiting values for the *b/t* ratio of plate elements in structural members, in terms $C/\sqrt{f_y}$, where *C* is a constant. These are dealt with in a subsequent chapter on local buckling.

3.0 POST-BUCKLING BEHAVIOUR AND EFFECTIVE WIDTH

3.1 Post-buckling Behaviour

Consider a rectangular plate with all four edges simply supported and subjected to uniform compression along *x*-direction (Fig. 1). When the compressive stress equals the critical buckling stress σ_{cr}, the central part of the plate, such as the strip *AB*, buckles. But the edges parallel to the *x*-axis cannot deflect in the *z*-direction and so the strips closer to these edges continue to carry the load without any instability. Therefore the stress distribution across the width of the plate in the post-buckling range becomes non-uniform with the outer strips carrying more stress than the inner strips as shown in Fig. 5(*a*). However, as described before, the transverse strips such as *CD* in Fig. 1 continue to stretch and support the longitudinal strips. This ensures the stability of the plate in the post-buckling range. Increasing the axial displacement of the plate will cause an increase in the lateral displacement. When the edge stresses approach and equal the yield stress of the material, the plate deflection would be vary large and the plate, eventually, can be considered to have failed when the stresses in the edge strips reach the yield stress of the material.

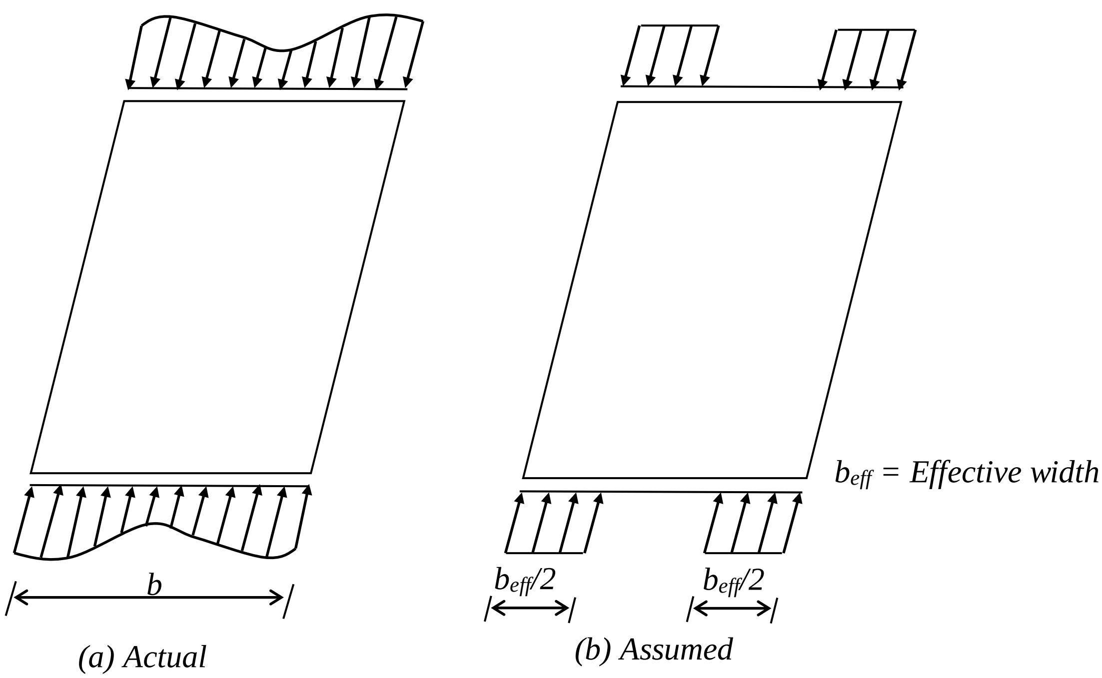

Fig. 5 Actual and Assumed Stress Distribution in the Post-buckling

3.2 Effective Width

To calculate the load carrying capacity of the plate in the post-buckling range, the concept of effective width is used. The concept was first proposed by von Karman. He realized that as the plate is loaded beyond its elastic buckling load, the central part such as strip AB deflects thereby shedding the load to the edge strips. Therefore, the non-uniform stress distribution across the width of the buckled plate, can be replaced by a uniform stress blocks of stress equal to that at the edges, over a width of $b_{eff}/2$ on either side where b_{eff} is called the *effective width* of the plate. This effective width can be calculated by equating the non-uniform stress blocks and the uniform stress blocks.

The shape of the non-uniform stress block depends on the load and support conditions. Therefore a number of formulae are available for calculating the effective width, each catering to a particular geometry of the plate. For the plate simply supported on all four sides, as the load is increased beyond the critical buckling load, the stress block becomes more and more non-uniform. When the stress at the outer strips reaches the yield stress, the corresponding effective width can be calculated using Winter's formula

$$\frac{b_{eff}}{b} = \sqrt{\left(\frac{\sigma_{cr}}{f_y}\right)\left[1 - 0.22\sqrt{\left(\frac{\sigma_{cr}}{f_y}\right)}\right]} \qquad (8)$$

The yield stress f_y, multiplied by the effective width gives the ultimate strength of the plate approximately.

3.3 Plates with Initial Imperfections

The perfectly flat plate described above represents an ideal condition. In practice, plates have initial imperfections, which (for simplicity of calculations) are normally assumed to be similar to the buckled shape. The behaviour of practical plates is broadly similar to the post-buckling behaviour of perfectly flat plates. However, the stresses across the width are non-uniform right from the beginning and so the concept of effective width can be applied to them even before the onset of elastic buckling. Other aspects of the behaviour of plates with initial imperfections such as stiffness and strength will be described in subsequent sections.

4.0 STABILITY AND ULTIMATE STRENGTH OF PLATES

4.1 Stability of Plates

It is interesting to compare the stability of a column and a plate. In the case of an ideal column, as the axial load is increased, the lateral displacement remains zero until the attainment of the critical buckling load (Euler load). If we plot the axial load versus lateral displacement, we will get a line along the load axis up to $P = P_{cr} = P_e$ (Fig. 7). This is called the fundamental path. When the axial load becomes equal to the Euler buckling load, the lateral displacement increases indefinitely at constant load. This is the secondary path, which bifurcates from the fundamental path at the buckling load. The secondary path for column represents neutral equilibrium. For practical columns, which have initial imperfections, there is a smooth transition from the stable to neutral equilibrium paths as shown by the dashed line in Fig. 6(*a*).

The fundamental path for a perfectly flat plate is similar to that of an ideal column. At the critical buckling load, this path bifurcates into a secondary path as shown in Fig. 6(*b*). The secondary path reflects the ability of the plate to carry loads higher than the elastic critical load. Unlike columns, the secondary path for a plate is stable. Therefore, elastic buckling of a plate need not be considered as collapse. However, plates having one free edge and simply supported along the other edges (outstands), have very little post-buckling strength.

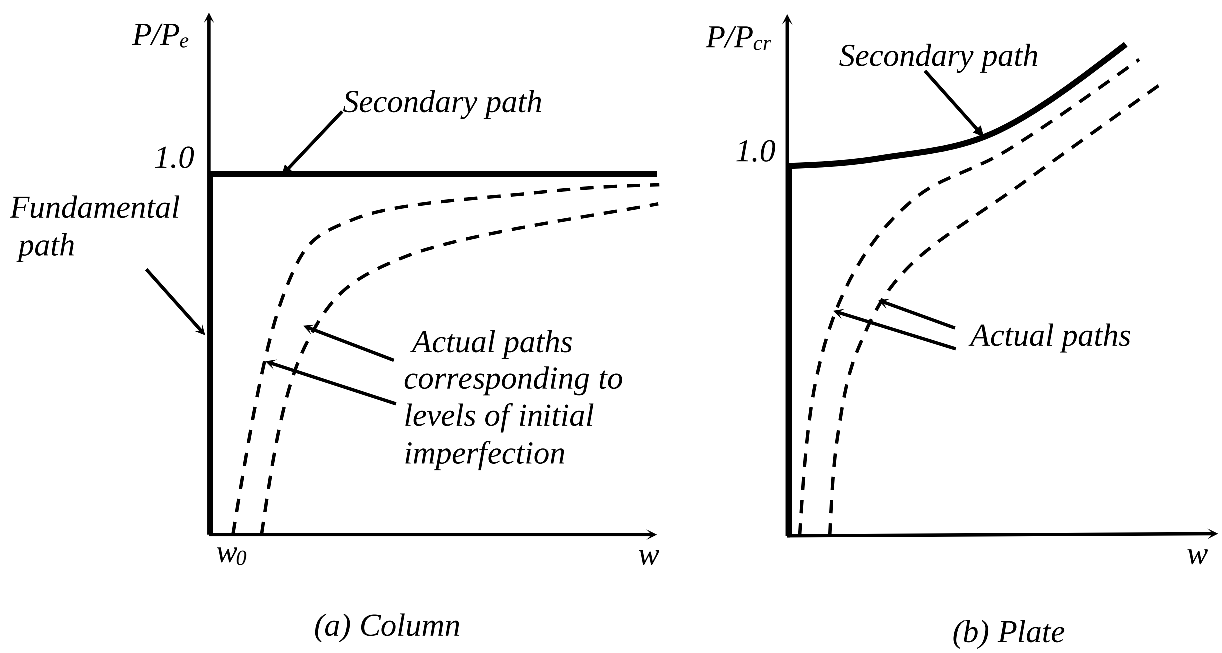

Fig. 7 Load versus Out-of-plane Displacement Curves

Actual failure load of the columns and plates are reached when the yielding spreads from the supported edges triggering collapse and thereafter the unloading occurs.

The axial stiffness of an ideally square flat plate drops suddenly from *EA* (where *A* is the cross-sectional area and E the elastic modulus) to a smaller value (nearly *AE/2*) after buckling and remains relatively constant thereafter, as shown by the load-axial deformation curve in Fig. 8. In the case of practical plates there is a gradual loss of stiffness as shown by the dashed line in Fig. 8.

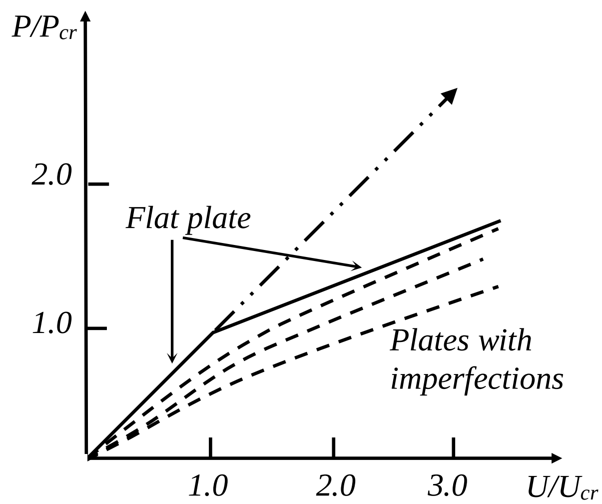

Fig. 8 Load versus Axial Deformation Diagram

4.2 Strength of Plates

Plate strength curves can also be constructed similar to the column strength curves (Fig. 9). In the case of ideal columns with low slenderness (*i.e.* stocky columns), failure is expected by squashing at the yield stress. On the other hand, if the ideal column is slender, failure will be by buckling at or near the Euler load. Tests on practical columns indicate that failure always occurs below the failure load of an ideal column of the same slenderness. If the column is stocky, then the yield stress provides an upper bound and if the column is slender then the buckling stress provides an upper bound. Also the scatter in the test results is considerable particularly in the range of intermediate slenderness ratios ($\sqrt{f_y/\sigma_e} = 1.0$).

In the case of a flat plate simply supported on all four sides, we can expect failure by squashing if the *b/t* ratio is less than the limiting value given by equation (7). Similarly, for *b/t* ratios larger than the limiting value, failure after buckling at the critical buckling stress may be expected. However, tests on practical plates indicate that for large *b/t* ratios, the failure stress is substantially greater than the critical buckling stress. This is due to the post-buckling behaviour, which is unique to plates. The load from the middle strips gets transferred to the edges and the plate continues to carry higher load in stable post-buckling range, until the edges reach the yield stress as described in the previous sections. As with columns, the scatter in the test results is considerable in the range of intermediate *b/t* ratios (at $f_y/\sigma_{cr} = 1.0$).

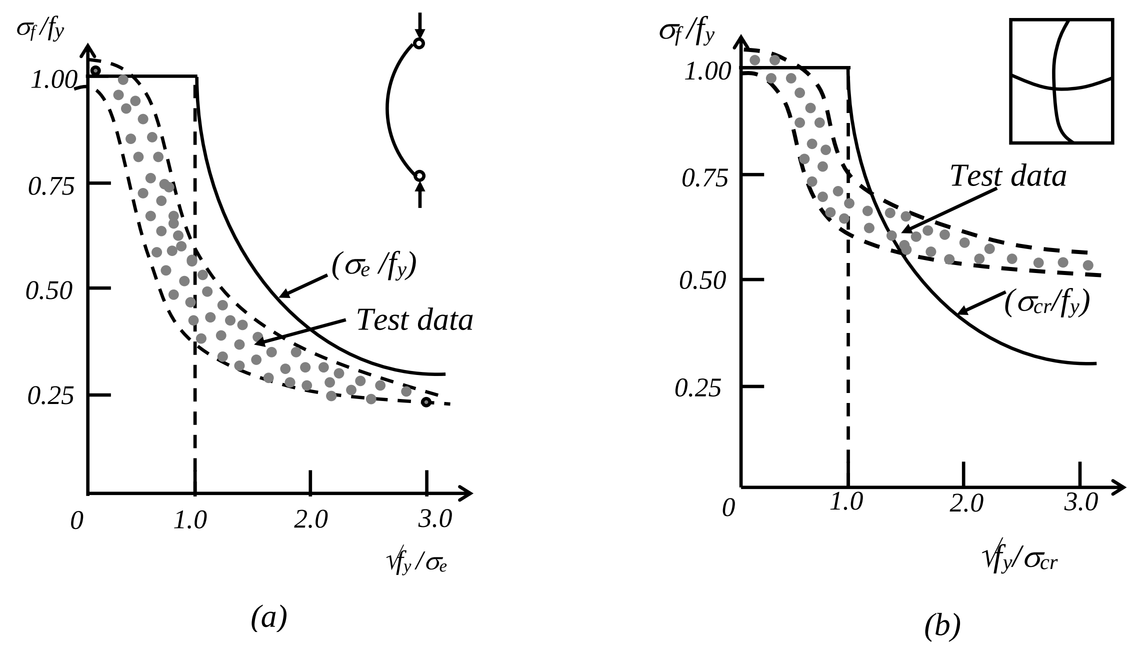

Fig. 9 Column and Plate Strength Curves

5.0 BUCKLING OF WEB PLATES IN SHEAR

Rectangular plates loaded in shear such as web plates in a plate girder, also tend to buckle. Consider a plate loaded in shear in its own plane as shown in Fig.10. A square element in the plate (Fig.10), whose edges are oriented at 45^o to the plate edges, experiences tensile stresses on two opposite edges and compressive stresses on the other two edges. The compressive stress can cause local buckling and as a result the plate develops waves perpendicular to them.

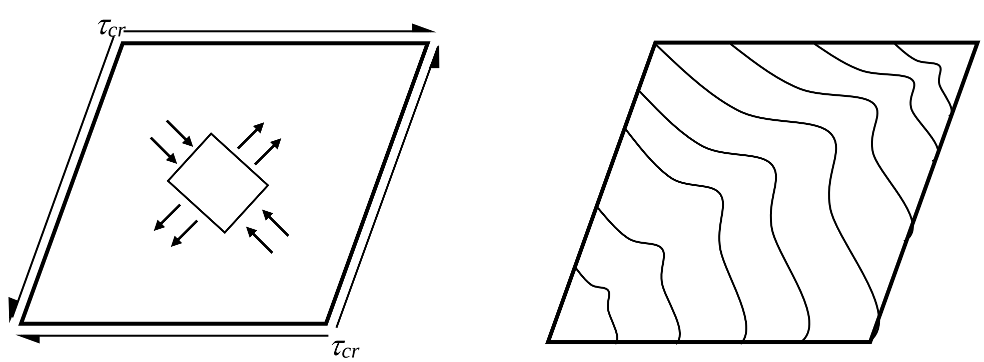

Fig. 10 Shear buckling of a plate

The critical shear stress at which this form of buckling occurs is given by the same formula as that for plate buckling under compression, except that the value for the buckling coefficient *k* is different (Table 1). The buckling coefficient varies with the aspect ratio *a/b* and increases from *5.34* for an infinitely long panel to *9.34* for a square panel. The values given in Table 1 are for the square panels.

Plates buckled in shear also can support additional loads. If we draw imaginary diagonals on the plate, the diagonal which gets loaded in compression buckles and cannot support additional load. However, the diagonal in tension continues to take more load and the plate becomes like a triangular truss with only tension diagonals. This is called ***tension field action.***

Formulae based on theoretical and experimental work have been produced which are of the form:

$$\tau' = \tau_{cr} + \frac{\tau_y \sqrt{3}}{2}\left[\frac{1-\tau_{cr}/\tau_y}{\sqrt{1+(a/d)^2}}\right] \quad \text{when } \tau_{cr} < \tau_y \text{ and } (a/d) < 3.0 \tag{9}$$

$$\tau' = \tau_{cr} \quad \text{when } \tau_{cr} \geq \tau_y \text{ and } (a/d) \geq 3.0$$

The first term represents the critical stress of an ideally flat plate and the second term, the post-buckling reserve due to the tension field action. In practical webs, the post-buckling reserve due to tension field may be several times the critical stress τ_{cr}. Web panels are also less sensitive to usual imperfections than compression flanges.

The critical load for four-side supported plate subjected to various stress combinations is given by the following interaction formula:

$$\left(\frac{\sigma}{\sigma_{cr}}\right) + \left(\frac{\sigma_b}{\sigma_{b,cr}}\right)^2 + \left(\frac{\tau}{\tau_{cr}}\right)^2 = 1 \tag{10}$$

where, σ, σ_b and τ are the applied axial compressive, maximum bending compressive and shear stress respectively and σ_{cr}, $\sigma_{b,cr}$ and τ_{cr} are the corresponding critical stresses.

6. 0 CONCLUDING REMARKS

Like columns, plates also undergo instability and buckle under compressive and shear stresses. The critical buckling load for a plate depends upon its width-thickness ratio and support conditions. However, unlike columns, the post-buckling behaviour of plates is stable and plates will continue to carry higher loads beyond their elastic critical loads. This post-buckling range is substantial in the case of plates with high width-thickness ratios (slender plates).

In the post-buckling range, the stress distribution is non-uniform and the plate fails when the maximum stress at the supported edge in the post-buckling range reaches the yield stress. So the concept of effective width is used to calculate the strength of the plate. This concept replaces the non-uniform stress distribution by an equivalent uniform stress equal to the stress at the edges over a reduced effective width (b_{eff}) of the plate.

In practical plates having initial imperfections, the stress distribution is non-uniform right from the start and so the effective width concept can be used even before the inception of buckling. Practical plates also posses post-buckling strength and empirical formulas are available to estimate their ultimate strength. However, the critical buckling strength of three side simply supported plates such as outstands is quite small and their post-buckling strength is also negligible.

7.0 REFERENCES

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